MAT 1341 Partial Exam I

March 4, 2019 Length: 80 minutes.
Professor:

1	D
2	F
3	F
4	
5	
6	
Total	

Family name:______________________

First name::______________________

Student number:____________________

PLEASE CAREFULLY READ THESE INSTRUCTIONS::

1. You have 80 minutes to write this exam.
2. You are not allowed to consult your notes or any books. Calculators, phones, and other electronic devices are not allowed.
3. Carefully read each question and **answer all questions in the space provided for this purpose.** For questions 4 to 6, you can use the back of the pages if necessary, but do not forget to indicate it to the T.A!
4. Questions 1 to 3 are multiple choice questions, each worth 1 point. No partial credit will be awarded. You must indicate the method you used to select the correct answer; unjustified answers will not be given credit.
5. Questions 4 to 6 are to be developed. They are each worth 6 points. **You must justify and write your answers correctly to get all possible points.**
6. Good luck! Bonne chance!

1. For a homogeneous system of 2019 equations with 269 unknowns, which of the following statements is true?

A. The system may be inconsistent
B. The system is never inconsistent and never has a unique solution
C. The system is never inconsistent and always has a unique solution
D. The system is never inconsistent and can have a unique solution
E. The system is never inconsistent and has a solution with 1750 parameters
F. The system has at least 1750 solutions

Solution:

A Homogeneous system is always consistent
may have
(i) unique solution or
(ii) infinitely many solutions

The system is never inconsistent (always consistent)
and can have a unique solution.

Note that: $0 \leq \text{rank}(A) \leq 269$

$$\begin{array}{c} \overbrace{\qquad\qquad}^{269} \\ 2019\left\{\left[\begin{array}{c|c} \qquad\qquad & 0 \\ & \vdots \\ & 0 \end{array}\right]\right. \end{array}$$

* If rank(A) = 269 (# of variables),
then we have unique solution

* If rank(A) < 269, then we have
infinitely many solutions.

Answer: D

2. Which of the following sets are bases of $\mathbb{R}^4$?

(1) $\{(1,0,0,0), (4,2,0,0), (1,4,0,0), (5,6,1,0)\}$

(2) $\{(0,-1,2,3), (0,3,3,2), (4,2,0,0)\}$

(3) $\{(-1,3,-5,1), (1,-2,4,2), (2,0,4,3), (5,1,9,4), (4,2,0,0)\}$

A. All three are bases.
B. Only (1) is a basis.
C. Only (2) is a basis.
D. (1) and (2) are both bases.
E. (1) and (3) are both bases.
F. None of these three are bases of $\mathbb{R}^4$

Solution: We know that $\dim(\mathbb{R}^4)=4$.
option (2) is too small option (3) is too big.
We need to check option(1) if it is L.I. We can
do it by solving $[A|0]$.

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 5 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 4 & 1 & 5 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{rank}(A) = \text{rank}([A|0]) = 3$

In addition, $\text{rank}(A)=3 < \#$ of columns $=4$

Hence, this system $[A|0]$ has infinitely many solutions $\Rightarrow$ option (1) is L.D. Consquently option (1) cannot be a basis of $\mathbb{R}^4$

Answer: F

3. Find the value of s for which $(1,-4,s)$ belongs to Span $\{(1,2,1),(2,-2,6),(4,14,0)\}$

A. $s=0$
B. $s=1$
C. $s=2$
D. $s=3$
E. $s=4$
F. $s=5$

Solution: Set $b=(1,-4,s)\in \text{Span}\{(1,2,1),(2,-2,6),(4,14,0)\}$

$\Longleftrightarrow [A|b]$ is consistent, where

$$A=\begin{bmatrix}1&2&4\\2&-2&14\\1&6&0\end{bmatrix}$$

We need to solve $[A|b]$ (consistent)

$$\boxed{A|b}=\left[\begin{array}{ccc|c}1&2&4&1\\2&-2&14&-4\\1&6&0&s\end{array}\right]\sim\left[\begin{array}{ccc|c}1&2&4&1\\1&-1&7&-2\\1&6&0&s\end{array}\right]\sim\left[\begin{array}{ccc|c}1&2&4&1\\0&3&-3&3\\0&-4&4&1-s\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&2&4&1\\0&1&-1&1\\0&-4&4&1-s\end{array}\right]\sim\left[\begin{array}{ccc|c}1&2&4&1\\0&1&-1&1\\0&0&0&5-s\end{array}\right]$$ consistent

$\Rightarrow\quad 5-s=0 \Rightarrow s=5$

Answer: F

4. Consider the vector space $\mathbb{F}([-1,1]) = \{f \mid f\colon [-1,1] \to \mathbb{R}\}$ of functions, defined from [-1, 1] to $\mathbb{R}$, with the standard operations. Recall that the zero vector of $\mathbb{F}([-1,1])$ is the function that has the value 0 for all $x \in [-1,1]$.

Define three functions f, g and h de $\mathbb{F}([-1,1])$ by

$$f(x) = x\,, \quad g(x) = 1 - 2x + x^2\,, \quad \text{and} \quad h(x) = 1 + x^3\,.$$

Now let, $W = \text{Span}\,\{f,\, g\,, h\}$

a) Show that f, g and h are linearly independent.

b) Find the base of W and determine the dimension of W.

c) If $k \in \mathbb{F}([-1,1])$ is the function defined by $k(x) = 2 + x^2 + x^3$, for $x \in [-1,1]$, show that $k \in W$.

d) What is the dimension of the subspace $Y = \text{Span}\,\{\,f,\, g\,, h,\, k,\}$?
(You must justify your answers)

Solution: $W = \text{span}\{f, g, h\}$

ⓐ $af(x) + bg(x) + ch(x) = 0 \Rightarrow a(x) + b(1-2x+x^2) + c(1+x^3) = 0$

$\Rightarrow ax + b - 2bx + bx^2 + c + cx^3 = 0$

$\Rightarrow cx^3 + bx^2 + (a-2b)x + b + c = 0$

$$\begin{cases} c = 0 \\ b = 0 \\ a - 2b = 0 \\ b + c = 0 \end{cases} \Rightarrow \begin{cases} c = 0 \\ b = 0 \\ a = 2b = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

we conclude that $\{f, g, h\}$ is L.I

ⓑ Since $\{f, g, h\}$ is L.I and $W = \text{span}\{f, g, h\}$

$\Rightarrow \{f, g, h\}$ is a base of W. $\dim(W) = 3$

(c) We have $K(x) = 2 + x^2 + x^3 = (2x - 2x) + 2 + x^2 + x^3$

$$= 2x + 1 - 2x + x^2 + 1 + x^3$$

$$= 2(x) + (1 - 2x + x^2) + (1 + x^3)$$

$$= 2f(x) + 1g(x) + 1h(x)$$

$$\Rightarrow K(x) \in \text{span}\{f, g, h\}$$

(d) Since $K \in W = \text{span}\{f, g, h\}$, $\text{span}\{f, g, h\} = \text{span}\{f, g, h, k\}$. Therefore $\dim(W) = \dim(Y) = 3$.

5. Consider the following linear system:

$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 1 \\ 2x & + & y & + & az & = & 2 \\ 3x & + & 4y & + & az & = & c \end{array}$$

a) Calculate the rank of the coefficient matrix A and the rank of the augmented matrix $[A|b]$, in function of the parameters a and c.

b) Deduce the values of a and c for which the system has

(*i*) no solutions
(*ii*) infinitely many solutions
(*iii*) unique solution.

c) In case b (ii) above, find the general solution of the system and give a complete geometrical description of the set of solutions.

Solution (a) We need to solve the linear system

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1 & a & 2 \\ 3 & 4 & a & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & a-4 & 0 \\ 0 & 1 & a-6 & c-3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -a+4 & 0 \\ 0 & 1 & a-6 & c-3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -a+4 & 0 \\ 0 & 0 & -2a+10 & 3-c \end{array}\right]$$

$$\text{rank}(A) = \begin{cases} 2 & \text{if } a = 5 \\ 3 & \text{if } a \neq 5 \end{cases}$$

$$\text{rank}([A|b]) = \begin{cases} 2 & \text{if } a = 5 \text{ and } c = 3 \\ 3, & \text{otherwise.} \end{cases}$$

(b) (i) No solutions: $\Rightarrow \operatorname{rank}(A) < \operatorname{rank}([A|b])$

$\Leftrightarrow a = 5$ and $c \neq 3$

(ii) Infinitely many solutions

$\operatorname{rank}(A) = \operatorname{rank}([A|b])$

and

$\operatorname{rank}(A) < \#$ of columns

$\Leftrightarrow a = 5$ and $c = 3$

(iii) a unique solution

$\operatorname{rank}(A) = \operatorname{rank}([A|b])$

and

$\operatorname{rank}(A) = \#$ of columns

$\Leftrightarrow a = 5$ and for all c

(3) In case (ii) Infinitely many solutions: $(a=5,\ c=3)$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1 and x_2 are leading variables
x_3 is non-leading variable $(x_3 = s)$

$x_1 = -3s + 1$
$x_2 = s$
$x_3 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3s+1 \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Geometric description this is a line in $\mathbb{R}^3$ through $(1,0,0)$ with direction vector $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

6. In each case, indicate in the corresponding box if the statement below is (always) true or can be false.

- If you feel that the statement may be wrong, give an explicit example that it is false.

- If you feel that the statement is (always) true, you must justify it with a clear explanation.

I. A system of 10 linear equations with 15 unknowns is consistent with many infinitely solutions.

$$\begin{array}{cc} & \overbrace{}^{\text{15 variables}} \\ \begin{array}{c}10\\ \text{equations}\end{array} & \left\{ \left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \\ 0 & \cdots & 0 & 1 \end{array}\right] \right. \end{array}$$

ANSWER $\boxed{\text{False}}$

Last row: degenerate equation $\Rightarrow$ inconsistent system

II. If the coefficient matrix A of a homogeneous linear system of 8 equations with 7 unknowns such that $\text{rank}(A) = 7$, then the system has a unique solution.

$$\begin{array}{cc} & \overbrace{}^{\text{7 variables}} \\ \begin{array}{c}8\\ \text{equations}\end{array} & \left[\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{array}\right] \end{array}$$

ANSWER $\boxed{\text{True}}$

* homgeneous system is always consistent $(\text{rank}(A) = \text{rank}([A|\vec{0}]))$

* Since $\text{rank}(A) = \#$ of variables $= 7 \Rightarrow$ the system has a unique solution.

III. If the coefficient matrix of a consistent linear system has a column of zeros, then the system has many infinitely solutions.

* This linear system is consistent

ANSWER $\boxed{\text{True}}$

$\Rightarrow \text{rank}(A) = \text{rank}([A|b])$ (1)

* This linear system has a column of zeros

$\Rightarrow \text{rank}(A) < \#$ of columns (2)

From (1) and (2), we conclude that this linear system has many infinitely solutions

IV. Consider two vectors u and v from a vector space V.

a) Give the definition of "$\{u, v\}$ is ~~an~~ linearly independent set."

$\{u, v\}$ is L.I iff $au + bv = 0 \Rightarrow a = b = 0$

b) If $\{u, v\}$ is a base of $\mathbb{R}^2$, then $\{u - v, v\}$ is linearly independent.

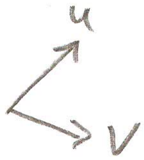
Method 1

ANSWER: True

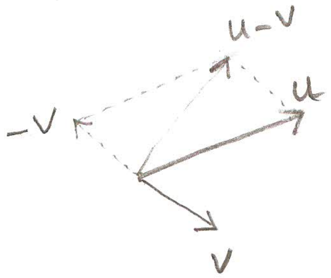
$\{u, v\}$ is a base of $\mathbb{R}^2 \Rightarrow \{u, v\}$ is L.I

$\Rightarrow$ u and v are not collinear. $(u \neq kv)$ or

$v \neq ku$ for some $k \in \mathbb{R}$. For example



Now, to prove that $\{u-v, v\}$ is L.I, we can just prove that $u-v$ and v are not collinear



$u-v$ and v are not collinear $\Rightarrow \{u-v, v\}$ is L.I

Method 2

$\{u, v\}$ is L.I $\Rightarrow$ $au + bv = 0 \Rightarrow a = b = 0$

$\Rightarrow au - av + bv = 0$

$\Rightarrow a(u-v) + bv = 0 \Rightarrow a = b = 0$

Hence, $\{u-v, v\}$ is L.I